4.3 Linearly Independent Sets; Bases

In this section, we generalize the notions of linearly independent sets and bases to vector spaces. The definition and results are almost identical but in a more general setting.

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solution, $c_1 = 0, \ldots, c_p = 0$.
The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, that is, if there are some weights, $c_1, \ldots, c_p$, not all zero, such that (1) holds. In such a case, (1) is called a linear dependence relation among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

Theorem 4. An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Definition Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a **basis** for H if
(i) $\mathcal{B}$ is a linearly independent set, and
(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \operatorname{Span} \mathcal{B}$$

Example 1. Determine which sets in the following are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

(1) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

Consider the matrix whose columns are the given vectors.

$$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the matrix has 3 pivot positions.

Thus the columns form a basis for $\mathbb{R}^3$.

(2) $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$

Since the zero vector is in the given set, the set cannot be linearly independent thus cannot be a basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ -3 & 9 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix has a pivot position in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

Example 2. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 3y + z = 0$. [Hint: Think of the equation as a "system" of homogeneous equations.]

ANS: Let $A = [1 \quad 3 \quad 1]$. Then the given plane $x+3y+z=0$ is the same as $A\vec{v} = \vec{0}$, where $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

So we need to find Nul A.

Then $x = -3y - z$ with y, z as free varibles.

i.e.

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y-z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and a basis for Nul A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

The Spanning Set Theorem

Theorem 5. The Spanning Set Theorem
Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in a vector space V, and let $H = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.
a. If one of the vectors in S-say, $\mathbf{v}_k$-is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
b. If $H \neq \{0\}$, some subset of S is a basis for H.

Example 3. Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} : s \text{ in } \mathbb{R}\right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$= \operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

$$\begin{bmatrix}s\\s\\0\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

a line in $\mathbb{R}^3$ a plane in $\mathbb{R}^3$

Question $H \neq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

We observe that $\vec{v}_1, \vec{v}_2$ are not in H. so $\{v_1, v_2\}$ cannot be a basis for H.

The Row Space

If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the **row space of** A and is denoted by Row A.

Remark:

1. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.
2. Since the rows of A are identified with the columns of A^T, we could also write $\mathrm{Col}\, A^T$ in place of Row A.

Bases for Nul A, Col A, and Row A

Theorem 6. The pivot columns of a matrix A form a basis for $\mathrm{Col}\, A$.

Theorem 7. If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Example 4. Assume that A is row equivalent to B. Find bases for Nul A, $\mathrm{Col}\, A$, and Row A.

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns for A are columns 1, 3, 5. Thus a basis for Col A by Thm 6 is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$

For Nul A, we need to solve $A\vec{x} = \vec{0}$. From the information of B. we have

$$\begin{cases} x_1 = -2x_2 - 4x_4 \\ x_3 = \frac{7}{5}x_4 \\ x_5 = 0 \end{cases}.$$

So $\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}$.

Thus a basis for Nul A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$

A basis for Row A can be taken from the nonzero rows of B:

$$\{[1\ 2\ 0\ 4\ 5],\ [0\ 0\ 5\ -7\ 8],\ [0\ 0\ 0\ 0\ -9]\}.$$

Example 5. Consider the polynomials $\mathbf{p}_1(t) = 1 + t^2$ and $\mathbf{p}_2(t) = 1 - t^2$. Is $\{\mathbf{p}_1, \mathbf{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

ANS: Observe that $p_1(t)$ and $p_2(t)$ are not scalar multiples of each other.

So $\{p_1, p_2\}$ is a linearly independent set in $\mathbb{P}_3$